ON THE FLOW OF AN ELECTRICALLY CONDUCTING FLUID IN TUBES OF ARBITRARY CROSS-SECTION IN THE PRESENCE OF A MAGNETIC FIELD

(O TECHENII ELEKTROPROVODNOI ZHIDKOSTI V PRISUTSTVII MAGNITNOGO POLIA PO TRUBAM PROIZBOL'NOGO PROFILIA)

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S. A. REGIRER
(Vorkuta)

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One of the problems of magneto-hydrodynamics is the study of the flow of a conducting viscous fluid in a tube located in a magnetic field. Its solutions are known for plane and round cylindrical tubes [1-3]. The general problem treated below of the steady flow in an infinitely long tube of arbitrary cross-section reduces to the successive solution of two linear boundary value problems. A result of a similar nature was obtained earlier [4] for a limited class of externally applied magnetic fields.

The equations of magneto-hydrodynamics for an incompressible medium have the form [2]:

$$(\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p^* + \frac{\varkappa}{\rho}(\mathbf{H}\nabla)\mathbf{H} + \nu\nabla\mathbf{v} \qquad (1)$$

$$(\mathbf{v}\nabla)\mathbf{v} = (\mathbf{H}\nabla)\mathbf{H} + \beta\triangle\mathbf{H}, \quad \operatorname{div}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{H} = 0 \qquad (2)$$

where

$$p^* = p + {}^1/_2\varkappa H^2, \qquad \varkappa = \mu/4\pi, \qquad \beta = c^2/4\pi\sigma\mu \qquad (3)$$

the remaining notations are conventional.

We shall consider a cylindrical tube, the generating lines of which are parallel to the z-axis; the cross-section in the plane $z = 0$ is a singly connected region B, bounded by a piecewise smooth closed contour Σ.

In the case when the flow lines are parallel to the generating lines and the outside magnetic field does not vary along the z-axis, the system (1), (2) has a solution of the form

$$\mathbf{v}\{0, 0, v(P)\}, \qquad \mathbf{H}\{H_x(P), H_y(P), H_z(P)\}, \qquad p^* = p^*(P, z) \ (p \text{ in region } B)$$

where the unknown functions, obviously satisfy the system

$$\frac{\partial p^*}{\partial x} = \varkappa\left(H_x \frac{H_x}{\partial x} + H_y \frac{\partial H_x}{\partial y}\right), \qquad \frac{\partial p^*}{\partial y} = \varkappa\left(H_x \frac{\partial H_y}{\partial x} + H_y \frac{\partial H_y}{\partial y}\right) \tag{4}$$

$$\frac{\partial p^*}{\partial z} = \varkappa\left(H_x \frac{\partial H_z}{\partial x} + H_y \frac{\partial H_z}{\partial y}\right) + \eta \triangle v \qquad \left(\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{5}$$

$$\triangle H_x = 0, \quad \triangle H_y = 0, \quad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0, \quad 0 = H_x \frac{\partial v}{\partial x} + H_y \frac{\partial v}{\partial y} + \beta \triangle H_z \tag{6}$$

Denoting by primes the quantities related to region B' which is outside B, the boundary conditions for v and H may be formulated in the following manner:

$$v = 0, \quad H_\tau = H_\tau', \quad \mu H_n = \mu' H_n', \quad H_z = f(P) \text{ on contour } \Sigma \tag{7}$$

We shall also consider as given the values H_x', H_y' at infinity and the flux of fluid Q through a cross-section of the tube. In the general case we shall add to the System (4)-(7) the equations of the plane field in the outside stationary medium, which by virtue of the assumption of invariance of H' along the z-axis will have the form:

$$\triangle H_x' = 0, \quad \triangle H_y' = 0, \quad \frac{\partial H_x'}{\partial x} + \frac{\partial H_y'}{\partial y} = 0 \tag{8}$$

We shall introduce the vectorial potentials of a plane field, assuming

$$H_x = \frac{\partial A}{\partial y}, \quad H_x' = \frac{\partial A'}{\partial y}, \quad H_y = -\frac{\partial A}{\partial x}, \quad H_y' = -\frac{\partial A'}{\partial x} \tag{9}$$

Then from (6) and (8) we obtain Poisson's equations for A and A' in which the right hand sides will be the constant vector components of the stream density ω and ω' with opposite signs. The quantities ω and ω' are not determined from the solution of the derived equations and must be given.

Thus, the first step in the solution of the problem consists of finding H_x and H_y, proceeding from the system of two Poisson's equations with compatibility conditions at the boundary and the condition at infinity

$$\triangle A = -\omega, \quad \triangle A' = -\omega', \quad \frac{\partial A}{\partial n}\bigg|_\Sigma = \frac{\partial A'}{\partial n}\bigg|_\Sigma, \quad \mu \frac{\partial A}{\partial \tau}\bigg|_\Sigma = \mu' \frac{\partial A'}{\partial \tau}\bigg|_\Sigma, \quad A'_{r\to\infty}{}' = A_\infty'. \tag{10}$$

Note that in classical electrodynamics problems of similar nature for $\omega' = 0$ (the problem of the magnetic field of a current carrying conductor) and for $\omega = \omega' = 0$ (the problem of bodies with magnetic properties in a magnetic field) are often encountered.

With the help of (10), Equations (4) are easily transformed to the form

$$\frac{\partial p^*}{\partial x} = \varkappa\left[\frac{1}{2}\frac{\partial}{\partial x}(H_x^2 + H_y^2) - \omega H_y\right], \quad \frac{\partial p^*}{\partial y} = \varkappa\left[\frac{1}{2}\frac{\partial}{\partial y}(H_x^2 + H_y^2) + \omega H_x\right]$$

Hence

$$p + \mu H_z^2 / 8\pi = \varkappa\omega A + C(z)$$

where it is not difficult to establish that p and C depend linearly on z. Therefore in Equation (5) we shall assume

$$\partial p^* / \partial z = \partial p / \partial z = \mathrm{const}$$

Consequently, the second step in the solution of the problem consists of finding v, H_z and $\partial p^*/\partial z$ from linear system

$$\frac{\partial p^*}{\partial z} = \varkappa\left(H_x\frac{\partial H_z}{\partial x} + H_y\frac{\partial H_z}{\partial y}\right) + \eta\triangle v, \qquad H_x\frac{\partial v}{\partial x} + H_y\frac{\partial v}{\partial y} + \beta\triangle H_z = 0 \tag{11}$$

$$v|_\Sigma = 0, \qquad \int_B v\,dB = Q, \qquad H_z|_\Sigma = f(P) \tag{12}$$

Assuming

$$u_1 = v + \lambda\beta H_z, \qquad u_2 = v - \lambda\beta H_z \qquad (\lambda = \sqrt{\varkappa/\eta\beta})$$

we obtain symmetrical expressions with separable unknowns:

$$\frac{1}{\eta}\frac{\partial p^*}{\partial z} = \lambda\left(H_x\frac{\partial u_1}{\partial x} + H_y\frac{\partial u_1}{\partial y}\right) + \triangle u_1, \quad \frac{1}{\eta}\frac{\partial p^*}{\partial z} = -\lambda\left(H_x\frac{\partial u_2}{\partial x} + H_y\frac{\partial u_2}{\partial y}\right) + \triangle u_2 \tag{13}$$

$$u_1|_\Sigma = \lambda\beta f(P), \qquad u_2|_\Sigma = -\lambda\beta f(P), \qquad \int_B (u_1 + u_2)\,dB = Q \tag{14}$$

The solutions of this elliptical system, evidently, may usefully be represented in terms of functions of a complex variable.

In a particular case, namely if in the outside medium and in the fluid, currents do not flow in the direction of the z-axis, i.e. $\omega = \omega' = 0$, and also if the magnetic properties of both media are the same, i.e. $\mu = \mu'$, then System (10) has a trivial solution $H_x = H_x' = H_{x\infty} =$ const, $H_y = H_y' = H_{y\infty} =$ const. Then the coordinate system may be chosen so that the x-axis will be coincident with the direction of the plane vector $\{H_x, H_y, 0\}$, and from (11) the equations are obtained which were derived by Shercliff in [4].

$$\frac{\partial p^*}{\partial z} = \varkappa H_x \frac{\partial H_z}{\partial x} + \eta \triangle v, \qquad H_x \frac{\partial v}{\partial x} + \beta \triangle H_z = 0.$$

If the region B is rectangular it is expedient to apply Schwartz Christoffel transformations. For circular regions the solution is represented effectively in the form of trigonometric series.

In conclusion we shall note that all the previous considerations are readily applicable to the case of an infinitely long vertical tube with free heat convection at constant (including zero) vertical wall temperature gradient in addition to the forced flow.

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